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Inclusion Theorems and Order Summability

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INTRODUCTION

In the preceding paper "Fourier Effectiveness and Order Summability," referred to by an added 'I' in front of the formulas, e.g., "I (3.6)", we have introduced *order summability* $[g]$ and *monotone summability methods A*. The basic result, which motivates the present paper, states that A is Fourier-effective if and only if $A \supseteq L_1{}^*$, where $L_1{}^*$ denotes logarithmic order summability. This follows from two results of the preceding paper, namely, that I (3.6) is necessary for Fourier-effectiveness and that $L_1{}^*$ is Fourier-effective, in combination with the fact that I (3.6) implies (and is equivalent to) $A \supseteq L_1{}^*$. The latter result is independent of Fourier series and will be proven in this paper. Together with several implications concerning Fourier series it has already been mentioned in Section 6 of the preceding paper.

Naturally, we shall discuss the general inclusion $A \supseteq [g]$ for arbitrary A and g. Section 1 gives sufficient conditions on A and g, while Section 2 gives necessary conditions. They coincide for monotone methods A and well-behaved g (Section 3). As a rather general and typical example, we discuss, in Section 4, *Wiener-type methods* (w). There we also prove inclusions of the type $A \subseteq [g]$.

In the preceding paper the importance of the intersection $\bigcap A$, A Fourier-effective, has been pointed out. It is interesting to note that almost the same intersection is obtained by using Fourier-effective Wiener-type methods only. More generally, in Section 5, we characterize the intersection $\bigcap (w)$, $(w) \supseteq [g]$, which is almost $[g]$, again. Finally, in Section 6, we discuss $\bigcap N_p$, $N_p \supseteq [g]$, and $\bigcap M_p$, $M_p \supseteq [g]$, where N_p and M_p denote monotone Nörlund means, resp. monotone arithmetical means. The former intersection coincides with $\bigcap C_\epsilon$ ($\epsilon > 0$), and the latter with C_1 as long as $[g]$ is weak enough (like $L_1{}^*$) but not equivalent to ordinary convergence. The independence of g

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of these intersections is remarkable and shows the relative inflexibility of these methods as compared to the Wiener-type methods. The special case $N_p \supseteq L_1{}^*$ is related to the known results of Hille-Tamarkin and Karamata concerning Fourier-effective Nörlund means.

1. Sufficient Conditions for $[g] \subseteq A$

We consider summability methods $A = (a_{n\nu})$ in the sequence-to-sequence form, satisfying

$$\begin{aligned} &a_{n\nu} \to 0 \qquad (n \to \infty,\ \nu \text{ fixed}), \\ &A_n = \sum_{\nu=0}^{\infty} a_{n\nu} \text{ converges and } A_n \to 1 \qquad (n \to \infty). \end{aligned} \tag{1.1}$$

Theorem 1.1. *Suppose that $g(t)$ is defined on* $[0, 1)$ *and that* $g(t) \geqslant 0$. *Let A satisfy* (1.1) *and assume that a sequence of integers $\nu_n \geqslant 0$ exists such that*

$$\sum_{\nu < \frac{1}{2}\nu_n} (\nu + 1) \mid \Delta a_{n\nu} \mid = O(1), \qquad \sum_{\nu > 2\nu_n} (\nu + 1) \mid \Delta a_{n\nu} \mid = O(1), \tag{1.2}$$

$$\sum_{\frac{1}{2}\nu_n \leqslant \nu < \nu_n} (\nu_n - \nu) \left(1 + g\left(\frac{\nu + 1}{\nu_n + 1}\right)\right) \mid \Delta a_{n\nu} \mid = O(1), \tag{1.3}$$

$$\sum_{\nu_n < \nu \leqslant 2\nu_n} (\nu - \nu_n) \left(1 + g\left(\frac{\nu_n + 1}{\nu + 1}\right)\right) \mid \Delta a_{n\nu} \mid = O(1). \tag{1.4}$$

Then $[g] \subseteq A$.

The proof depends upon the formula ($\sigma_n = 1/(n + 1) \sum_{\nu=0}^{n} s_\nu = S_n/(n + 1)$):

$$\begin{aligned} \sum_{\nu=0}^{\infty} a_{n\nu}(s_\nu - \sigma_k) = &\left(\sum_{\nu < \frac{1}{2}k} + \sum_{\nu > 2k}\right) (\nu + 1)(\sigma_\nu - \sigma_k)\, \Delta a_{n\nu} \\ &+ \sum_{\frac{1}{2}k \leqslant \nu \leqslant 2k, \nu \neq k} (\nu - k) \left(\frac{S_\nu - S_k}{\nu - k} - \sigma_k\right) \Delta a_{n\nu}, \\ &\qquad (n \geqslant 0, \quad k \geqslant 0), \end{aligned}$$

which holds when $s_n \to s\ (C_1)$. (Note that $[g] \subseteq C_1$, that $S_\nu - (\nu + 1)\, \sigma_k = S_\nu - S_k - (\nu - k)\, \sigma_k$, and that $\nu a_{n\nu} \to 0$ for $\nu \to \infty$, n fixed, is a consequence of (1.2).)

If $\nu_n \to \infty$, then the assertion is immediate, with $k = \nu_n$. If $\nu_n = O(1)$, we put $k = 0$ and observe that $\sum_{\nu=0}^{\infty} (\nu + 1) \mid \Delta a_{n\nu} \mid = O(1)$. The proof for an arbitrary sequence $\{\nu_n\}$ follows from the arguments used in these two special cases.

It is natural to assume that

$$g(t) \uparrow \text{ for } t \uparrow, \qquad g(0) > 0, \tag{1.5}$$

and it will be convenient to define

$$g(t) = \frac{1}{t} g\left(\frac{1}{t}\right) \qquad \text{for} \qquad t > 1, \tag{1.6}$$

$$g^*(t) = tg(t) \qquad \text{for} \qquad t \neq 1, \quad t \geqslant 0. \tag{1.7}$$

(The assumption $g(0) > 0$ does not restrict the generality since $[g]$ is equivalent to $[g_\alpha]$, $g_\alpha(t) = \alpha + g(t)$, $\alpha \geqslant 0$. This observation will frequently be used in the following.)

If g satisfies (1.5), then (1.2), (1.3) and (1.4) can be rewritten as

$$\sum_{\substack{\nu=0 \\ \nu \neq \nu_n}}^{\infty} | \nu - \nu_n | g^*\left(\frac{\nu+1}{\nu_n+1}\right) | \Delta a_{n\nu} | = O(1). \tag{1.8}$$

(A special case of this condition was I (6.1).)

For monotone methods, (1.8) can be simplified. We have

THEOREM 1.2. *Let g satisfy* (1.5), *and let A be monotone. Suppose that*

$$\sum_{\nu=0}^{\nu_n} a_{n\nu} g\left(\frac{\nu}{\nu_n+1}\right) = O(1), \qquad \sum_{\nu=\nu_n+1}^{\infty} a_{n\nu} g\left(\frac{\nu_n+1}{\nu+1}\right) = O(1) \tag{1.9}$$

(ν_n *is as in* I (3.7)). *Then* $[g] \subseteq A$.

If A is triangular and $\nu_n = n$, then (1.9) *takes the simpler form*

$$\sum_{\nu=0}^{n} a_{n\nu} g\left(\frac{\nu}{n+1}\right) = O(1). \tag{1.10}$$

Proof. The left side of (1.8) equals

$$\sum_{\nu=0}^{\nu_n} a_{n\nu} g^*\left(\frac{\nu}{\nu_n+1}\right) + \sum_{\nu=\nu_n+1}^{\infty} a_{n\nu} g^*\left(\frac{\nu+1}{\nu_n+1}\right)$$
$$+ \sum_{\nu=0}^{\nu_n-1} a_{n\nu}(\nu - \nu_n)\left(g^*\left(\frac{\nu+1}{\nu_n+1}\right) - g^*\left(\frac{\nu}{\nu_n+1}\right)\right)$$
$$+ \sum_{\nu=\nu_n+2}^{\infty} a_{n\nu}(\nu - 1 - \nu_n)\left(g^*\left(\frac{\nu+1}{\nu_n+1}\right) - g^*\left(\frac{\nu}{\nu_n+1}\right)\right),$$

where the last two terms are $\leqslant 0$. Therefore, (1.8) is a consequence of (1.9).

Remarks. 1. Conditions (1.8) and (1.9) resp. (1.10) are different, as can be seen in the special case $A = C_1$.

2. It follows from

$$\sum_{m=1}^{k} \frac{1}{m} \sum_{\nu=k-m}^{k+m-1} a_{n\nu} = \sum_{\nu=0}^{2k-1} a_{n\nu} \sum_{m=\max(k-\nu,\nu-k+1)}^{k} \frac{1}{m} \qquad (n \geqslant 0, k \geqslant 1)$$

that I (3.6), for positive (regular) A, is equivalent to

$$\sum_{\nu=0}^{k-1} a_{n\nu} \left(1 + \log \frac{k}{k-\nu}\right) + \sum_{\nu=k}^{\infty} a_{n\nu} \left(1 + \log \frac{\nu+1}{\nu+1-k}\right) = O(1),$$
$$(n \geqslant 0, k \geqslant 1) \qquad (1.11)$$

which is, for $k = \nu_n + 1$, condition (1.9) with $g(t) = 1 + \log 1/(1-t)$. Hence, *for monotone methods*, I (3.6) *is sufficient for* $A \supseteq L_1^*$. It is also necessary, since $A \supseteq L_1^*$ implies F_C-effectiveness of A which, in turn, implies I (3.6). We have used this result in Section 6 of the preceding paper. A more direct proof, based on Theorem 3.2, will be given at the end of Section 3.

2. Necessary Conditions for $[g] \subseteq A$

Let A be an arbitrary matrix and let $g(t) \geqslant 0$ for $t \in [0, 1)$. The summability $s_n \to s$ (A) involves the existence of $\sigma_n = \sum_{\nu=0}^{\infty} a_{n\nu} s_\nu$ (in some sense, e.g., ordinary convergence or C_1-summability) and $\sigma_n \to s$ $(n \to \infty)$. If $[g] \subseteq A$, then A is regular since $[g]$ is regular.

Clearly, the summability field of $[g]$ is a Banach space $\langle g \rangle$ with the norm

$$\|\{s_n\}\|_g = \sup_{0 \leqslant n, 0 \leqslant m \leqslant n} \left| \frac{\sigma_{nm}}{1 + g\left(\frac{m}{n+1}\right)} \right|, \qquad \sigma_{nm} = \frac{1}{n+1-m} \sum_{\nu=m}^{n} s_\nu ,$$

and the coordinates s_n depend continuously upon the sequence $\{s_\nu\}$. If $[g] \subseteq A$, then the linear mapping

$$\{s_n\} \to \{\sigma_n\}$$

takes sequences of $\langle g \rangle$ into the Banach space of convergent sequences, with norm

$$\|\{\sigma_n\}\| = \sup_{n \geqslant 0} | \sigma_n | = \|\{s_n\}\|_A .$$

Since σ_n, by Banach's limit theorem, depends continuously upon $\{s_n\}$, our mapping is closed and hence continuous. Thus, the inclusion $[g] \subseteq A$ implies the existence of a constant M such that

$$\|\{s_n\}\|_A \leqslant M \|\{s_n\}\|_g \qquad \text{for} \qquad \{s_n\} \in \langle g \rangle. \tag{2.1}$$

These arguments are standard, and (2.1) is actually the main condition involved in the inclusion. At the moment we do not know necessary and sufficient conditions for (2.1) in simple terms of $a_{n\nu}$ and g. We shall derive necessary conditions by constructing special sequences $\{s_n(k)\}$, depending upon a parameter k, which are uniformly $[g]$-bounded and, therefore, uniformly A-bounded. In Section 3 we shall see that for monotone methods the sufficient conditions of Section 1 coincide with the necessary conditions.

Our construction requires further conditions on g. Suppose that

$$g(t) = \frac{1}{1-t} \int_t^1 \tilde{g}(x)\, dx \qquad \text{for} \qquad t \in [0, 1),$$

where (2.2)

$$0 \leqslant \tilde{g}(t) \uparrow \quad \text{for} \quad t \uparrow (t \in [0, 1)), \qquad 0 < \int_0^1 \tilde{g}(t)\, dt < \infty.$$

Then (1.5) holds automatically, and we use (1.6) and (1.7) again. We also define

$$\tilde{g}(t) = \frac{1}{t^2} \tilde{g}\left(\frac{1}{t}\right) \qquad \text{for} \qquad t > 1. \tag{2.3}$$

THEOREM 2.1. *Let g satisfy* (2.2), *and let $A \supseteq [g]$ be arbitrary. Then for some constant $M > 0$,*

$$\left| \sum_{\nu=0}^{\infty} a_{n\nu}(k+1) \int_{\nu/(k+1)}^{(\nu+1)/(k+1)} \tilde{g}(t)\, dt \right| \leqslant M \qquad (n \geqslant 0, k \geqslant 0). \tag{2.4}$$

In particular, if $a_{n\nu} \geqslant 0$ ($n, \nu = 0, 1,\ldots$) then, for $n \geqslant 0$, $k \geqslant 0$,

$$\sum_{\nu=0}^{k} a_{n\nu} \tilde{g}\left(\frac{\nu}{k+1}\right) \leqslant M, \qquad \sum_{\nu=k+1}^{\infty} a_{n\nu} \tilde{g}\left(\frac{\nu+1}{k+1}\right) \leqslant M, \tag{2.5}$$

$$a_{nk}\, g\left(\frac{k}{k+1}\right) \leqslant M. \tag{2.6}$$

If (A *is regular and*) $\nu a_{n\nu} = o(1)$ ($\nu \to \infty$, *n fixed*), *then* (2.4) *is equivalent to*

$$\left| \sum_{\nu \neq k} (\nu - k)\, g^* \left(\frac{\nu+1}{k+1}\right) \Delta a_{n\nu} \right| \leqslant M' \qquad (n \geqslant 0, k \geqslant 0). \tag{2.7}$$

Proof. First we derive some simple inequalities and identities.

$$\tilde{g}(t) \leqslant g(t) \quad (t \in [0, 1)), \qquad \tilde{g}(t) \leqslant \frac{1}{t} g(t) \quad (t > 1), \tag{2.8}$$

$$\tilde{g}(t) \downarrow 0 \quad \text{for} \quad t \uparrow \infty \qquad (t > 1), \tag{2.9}$$

$$\int_1^t \tilde{g}(x)\, dx = \left(1 - \frac{1}{t}\right) g\left(\frac{1}{t}\right) = (t - 1)\, g(t) \qquad (t > 1).$$

Define $g_*(t) = t\tilde{g}(t) - (1 - t)\, g(t)$ $(t \geqslant 0, t \neq 1)$, and observe that

$$\int_t^1 g_*(x)\, dx = (1 - t)\, g^*(t) \quad (t \in [0, 1)), \qquad \int_0^1 g_*(x)\, dx = 0,$$

$$\int_1^t g_*(x)\, dx = (t - 1)\, g^*(t) \quad (t > 1),$$

$$\int_0^t g_*(x)\, dx = (t - 1)\, g^*(t) \quad (t \geqslant 0, t \neq 1),$$

$$\tilde{g}(t) - g_*(t) \text{ is bounded for } t \geqslant 0, \qquad t \neq 1 \tag{2.10}$$

in view of (2.2) and $(1 - t)\, \tilde{g}(t) = o(1)$ $(t \to 1 \pm 0)$.

The key to the proof is the following inequality:

$$\frac{1}{b - a} \int_a^b \tilde{g}(t)\, dt \leqslant 2g\left(\frac{a}{b}\right) \qquad \text{for} \qquad 0 \leqslant a < b. \tag{2.11}$$

There are three cases: (i) If $b \leqslant 1$, then

$$\frac{1}{b - a} \int_a^b \tilde{g}(t)\, dt \leqslant \frac{1}{1 - a} \int_a^1 \tilde{g}(t)\, dt = g(a) \leqslant g\left(\frac{a}{b}\right).$$

(ii) If $a < 1 < b$, then

$$\frac{1}{b - a} \int_a^b \tilde{g}(t)\, dt \leqslant \frac{1}{b - a} \int_{a/b}^{b/a} \tilde{g}(t)\, dt = \frac{2}{b}\, g\left(\frac{a}{b}\right) \leqslant 2g\left(\frac{a}{b}\right).$$

(iii) If $a \geqslant 1$, then

$$\frac{1}{b - a} \int_a^b \tilde{g}(t)\, dt \leqslant \frac{1}{b - 1} \int_1^b \tilde{g}(t)\, dt = \frac{1}{b}\, g\left(\frac{1}{b}\right) \leqslant g\left(\frac{a}{b}\right).$$

Depending upon an integer $k \geqslant 0$, we define a sequence

$$s_n(k) = (k + 1) \int_{n/(k+1)}^{(n+1)/(k+1)} \tilde{g}(t)\, dt \geqslant 0 \qquad (n \geqslant 0).$$

Obviously,

$$s_n(k) \to 0 \qquad (n \to \infty,\ k \text{ fixed});$$

therefore, $s_n(k) \to 0\ [g]$. Furthermore, by (2.11),

$$\sigma_{nm}(k) = \frac{k+1}{n+1-m} \int_{m/(k+1)}^{(n+1)/(k+1)} \tilde{g}(t)\, dt$$
$$\leqslant 2g\left(\frac{m}{n+1}\right) \leqslant 2\left(1 + g\left(\frac{m}{n+1}\right)\right)$$

for $0 \leqslant m \leqslant n$ and $k \geqslant 0$. If we apply (2.1) we obtain (2.4).

Next, we define

$$s_n{}^*(k) = (k+1) \int_{n/(k+1)}^{(n+1)/(k+1)} g_*(t)\, dt \qquad (n \geqslant 0,\ k \geqslant 0)$$

and note that

$$s_n(k) - s_n{}^*(k) \text{ is uniformly bounded} \qquad (n \geqslant 0,\ k \geqslant 0),$$

in view of (2.10). If A is regular, condition (2.4) is equivalent to

$$\left| \sum_{\nu=0}^{\infty} a_{n\nu}(k+1) \int_{\nu/(k+1)}^{(\nu+1)/(k+1)} g_*(t)\, dt \right| \leqslant M' \qquad (n \geqslant 0,\ k \geqslant 0). \quad (2.12)$$

If $\nu a_{n\nu} = o(1)$ ($\nu \to \infty$, n fixed), partial summation can be used, and (2.12) takes the form (2.7).

If $a_{n\nu} \geqslant 0$ ($n, \nu = 0, 1, \ldots$), (2.5) and (2.6) follow immediately from (2.4), because of (2.2) and (2.9).

3. Necessary and Sufficient Conditions for $[g] \subseteq A$

From Theorems 1.1 and 2.1 we obtain

Theorem 3.1. *Let g satisfy* (2.2), *and let A be monotone. Then $[g] \subseteq A$ is equivalent to each of the following requirements*:

Condition (2.7), *even with $k = \nu_n$*,

Condition (2.4), *even with $k = \nu_n$*,

Condition (2.5), *even with $k = \nu_n$, in conjunction with* (2.6), *even with $k = \nu_n$ and $k = \nu_n + 1$.*

Proof. In view of Theorem 2.1, we need only show that these requirements are sufficient.

Under the given assumptions on A and g, conditions (2.7) with $k = \nu_n$ and (1.8) are identical, and (2.7) is equivalent to (2.4) (by Theorem 2.1). It then follows from Theorem 1.1 that the first two requirements are sufficient. As to the third requirement, we show that (2.4) with $k = \nu_n$ is a consequence of (2.5) with $k = \nu_n$ and (2.6) with $k = \nu_n, \nu_n + 1$. We have

$$\sum_{\nu=0}^{\nu_n-1} a_{n\nu}(\nu_n + 1) \int_{\nu/(\nu_n+1)}^{(\nu+1)/(\nu_n+1)} \tilde{g}(t)\, dt$$

$$\leqslant \sum_{\nu=0}^{\nu_n-1} a_{n\nu} \tilde{g}\left(\frac{\nu+1}{\nu_n+1}\right) \leqslant \sum_{\nu=0}^{\nu_n-1} a_{n,\nu+1} \tilde{g}\left(\frac{\nu+1}{\nu_n+1}\right) \leqslant \sum_{\nu=0}^{\nu_n} a_{n\nu} \tilde{g}\left(\frac{\nu}{\nu_n+1}\right),$$

and this, together with a similar estimate of

$$\sum_{\nu=\nu_n+2}^{\infty} a_{n\nu}(\nu_n + 1) \int_{\nu/(\nu_n+1)}^{(\nu+1)/(\nu_n+1)} \tilde{g}(t)\, dt,$$

yields

$$\sum_{\nu=0}^{\infty} a_{n\nu}(\nu_n + 1) \int_{\nu/(\nu_n+1)}^{(\nu+1)/(\nu_n+1)} \tilde{g}(t)\, dt$$

$$\leqslant \sum_{\nu=0}^{\nu_n} a_{n\nu} \tilde{g}\left(\frac{\nu}{\nu_n+1}\right) + \sum_{\nu=\nu_n+1}^{\infty} a_{n\nu} \tilde{g}\left(\frac{\nu+1}{\nu_n+1}\right) + a_{n\nu_n} g\left(\frac{\nu_n}{\nu_n+1}\right)$$

$$+ a_{n,\nu_n+1} g\left(\frac{\nu_n+1}{\nu_n+2}\right).$$

The third requirement in Theorem 3.1 and Theorem 1.2 lead to the question whether the conditions (1.9) are also necessary and sufficient for $[g] \subseteq A$ in certain cases. If some $\epsilon > 0$ exists such that $\tilde{g}(t) \geqslant \epsilon g(t)$ as $t \to 1 - 0$, then it follows from (1.6) and (2.8) that the combined conditions (2.5) and (2.6) are equivalent to

$$\sum_{\nu=0}^{k} a_{n\nu} g\left(\frac{\nu}{k+1}\right) \leqslant M^*, \qquad \sum_{\nu=k+1}^{\infty} a_{n\nu} g\left(\frac{k+1}{\nu+1}\right) \leqslant M^* \quad (n \geqslant 0, k \geqslant 0) \tag{3.1}$$

(A regular and $a_{n\nu} \geqslant 0$ for $n, \nu = 0, 1, \ldots$; g satisfying (2.2)). In view of $\tilde{g}(t) = g(t) - (1 - t)\, g'(t)$, this is the case if

$$\limsup_{t \to 1-0} (1 - t) \frac{g'(t)}{g(t)} < 1. \tag{3.2}$$

Thus we have

THEOREM 3.2. *Let A be monotone, and let g satisfy* (2.2) *and* (3.2). *Then* (3.1), *even with $k = \nu_n$, is necessary and sufficient for* $[g] \subseteq A$.

Remark. For $g(t) = 1 + \log 1/(1-t)$ the conditions (2.2) and (3.2) are satisfied. It follows from Theorem 3.2 that for monotone A, $L_1{}^* \subseteq A$ if and only if (1.11), i.e., I (3.6) holds.

4. WIENER-TYPE METHODS

Let w be a function defined on $[0, 1)$ and satisfying

$$w(t) \geqslant 0, \qquad w(t)\uparrow \quad \text{for} \quad t\uparrow, \qquad \int_0^1 w(t)\,dt = 1. \tag{4.1}$$

Because of

$$\int_0^{n/(n+1)} w(t)\,dt \leqslant \frac{1}{n+1}\sum_{\nu=0}^{n} w\left(\frac{\nu}{n+1}\right) \leqslant \int_0^1 w(t)\,dt,$$

the transformation $W_n = 1/(n+1)\sum_{\nu=0}^{n} w(\nu/(n+1))\,s_\nu$ defines a monotone ($\nu_n = n$) and triangular summability method (w), a *Wiener-type method,* and we shall write $s_n \to s$ (w) when $W_n \to s$.

In this section we shall discuss conditions for the inclusion $[g] \subseteq (w)$, and also for $(w) \subseteq [g]$. Concerning the first of these relations, we derive from Theorems 1.2, 3.1 and 3.2 the following result.

THEOREM 4.1. *If g satisfies* (1.5), *then a sufficient condition for* $[g] \subseteq (w)$ *is*

$$\int_0^1 w(t)\,g(t)\,dt < \infty. \tag{4.2}$$

If g satisfies even (2.2), *then a necessary and sufficient condition for* $[g] \subseteq (w)$ *is*

$$\int_0^1 w(t)\,\tilde{g}(t)\,dt < \infty \tag{4.3}$$

or, equivalently,

$$\int_0^{1-0} (1-t)\,g(t)\,dw(t) < \infty. \tag{4.4}$$

If g satisfies (2.2) *and* (3.2), *then* (4.2) *is also necessary.*

Proof. Since w and g are nonnegative and nondecreasing, we have

$$\int_0^{n/(n+1)} w(t)\, g(t)\, dt \leqslant \frac{1}{n+1} \sum_{\nu=0}^{n} w\left(\frac{\nu}{n+1}\right) g\left(\frac{\nu}{n+1}\right) \leqslant \int_0^1 w(t)\, g(t)\, dt,$$

i.e., in the present case condition (4.2) is condition (1.10). A similar argument shows that (4.3) is condition (2.4) ($k = \nu_n$). It remains to show that (4.3) and (4.4) are equivalent, i.e., that (4.3), and also (4.4), implies

$$(1 - t)\, g(t)\, w(t) = o(1) \qquad \text{as} \qquad t \to 1 - 0. \tag{4.5}$$

(Note that

$$\int_0^{1-\epsilon} w(t)\, \tilde{g}(t)\, dt = -w(t) \int_t^1 \tilde{g}(x)\, dx \Big|_0^{1-\epsilon} + \int_0^{1-\epsilon} (1 - t)\, g(t)\, dw(t)$$

$$(0 < \epsilon < 1).)$$

Let (4.3) be satisfied. Then (4.5) follows from

$$\int_t^1 w(x)\, \tilde{g}(x)\, dx \geqslant w(t) \int_t^1 \tilde{g}(x)\, dx = (1 - t)\, g(t)\, w(t).$$

Let (4.4) be satisfied. We define a bounded, nondecreasing function

$$F(t) = \int_0^t (1 - x)\, g(x)\, dw(x), \qquad \text{for} \quad t \in [0, 1),$$

and observe that

$$w(t) - w(0) = \int_0^t \frac{1}{(1 - x)\, g(x)}\, dF(x) \qquad \text{for} \qquad t \in [0, 1). \tag{4.6}$$

Since $(1 - t)\, g(t) = \int_t^1 \tilde{g}(x)\, dx \downarrow 0$ as $t \uparrow 1$, and $\int_t^{1-0} dF(x) = o(1)$ as $t \to 1 - 0$, it follows that $w(t) - w(0) = o(1/(1 - t)\, g(t))$ $(t \to 1 - 0)$, which implies (4.5).

Remark. We see from this proof that (4.4) can be reformulated to yield the following result: If g satisfies (2.2), then $(w) \supseteq [g]$ if and only if w is of the form (4.6) with a bounded, nondecreasing F. Any such function F, apart from a constant factor (normalization), can actually occur.

Next, we turn to the inclusion $(w) \subseteq [g]$. Using the concept of a mean-value condition, we can employ standard arguments to discuss even the more general inclusion relation $A \subseteq [g]$ for a triangular and monotone A (see, e.g., [6]).

A triangular matrix $A = (a_{n\nu})$ satisfies the mean-value condition $M_K(A)$ if an inequality

$$\left|\sum_{\nu=0}^{m} a_{n\nu}s_\nu\right| \leqslant K \sup_{0\leqslant p\leqslant m}\left|\sum_{\nu=0}^{p} a_{p\nu}s_\nu\right| \qquad (0 \leqslant m \leqslant n)$$

holds, with K independent of m, n and $\{s_\nu\}$.

THEOREM 4.2. *Let A be triangular and monotone ($\nu_n = n$), and suppose that $M_K(A)$ holds. Then $A \subseteq [g]$ if $\epsilon > 0$, $\delta \in (0, 1)$ exist such that*

$$a_{nm} \geqslant \frac{\epsilon}{(n+1-m)\, g\left(\dfrac{m}{n+1}\right)} \quad \textit{for} \quad n \geqslant 0, \qquad \delta n \leqslant m \leqslant n. \quad (4.7)$$

Proof. Let $s_n \to 0$ (A). We have $A \subseteq C_1$ (see, e.g., [6, Theorem II.21]), and, hence, it is sufficient to show that

$$\frac{1}{n+1-m}\sum_{\nu=m}^{n} s_\nu = o\left(1 + g\left(\frac{m}{n+1}\right)\right) \qquad \text{for } n \to \infty, \text{ uniformly in } \delta n \leqslant m \leqslant n. \quad (4.8)$$

But $M_K(A)$ implies

$$\sup_{0\leqslant p\leqslant q\leqslant n}\left|\sum_{\nu=p}^{q} a_{n\nu}s_\nu\right| \to 0 \qquad \text{for} \qquad n \to \infty \quad (4.9)$$

[6, Theorem II.8, Lemma II.4], and (4.8) follows from

$$\sum_{\nu=m}^{n} \frac{1}{a_{n\nu}} a_{n\nu}s_\nu = \sum_{\nu=m}^{n-1}\left(\Delta \frac{1}{a_{n\nu}}\right)\sum_{\mu=m}^{\nu} a_{n\mu}s_\mu + \frac{1}{a_{nn}}\sum_{\mu=m}^{n} a_{n\mu}s_\mu\,,$$

by (4.7) and (4.9).

It remains to give conditions which ensure $M_K(w)$. We have $M_1(w)$ if $w(\nu/(n+2))/w(\nu/(n+1)) \downarrow$ for $\nu \uparrow$, $0 \leqslant \nu \leqslant n$ [6, Lemma II.5, Theorem II.16]; therefore, $M_1(w)$ is a consequence of $w(ax)/w(bx) \downarrow$ for $x \uparrow$ whenever $0 \leqslant a < b \leqslant 1$ $(0 \leqslant x < 1)$. If w' exists, then

$$\frac{xw(bx)}{w(ax)}\frac{d}{dx}\frac{w(ax)}{w(bx)} = \left(\frac{axw'(ax)}{w(ax)} - \frac{bxw'(bx)}{w(bx)}\right).$$

Thus we have

THEOREM 4.3. *Let (w) be a Wiener-type method with $w(0) > 0$. Suppose that $w'(t)$, $t \in [0, 1)$, exists and that*

$$t\frac{w'(t)}{w(t)} \uparrow \qquad \text{for} \quad t \uparrow. \quad (4.10)$$

Then $M_1(w)$ *holds. Furthermore, if* g *satisfies* (1.5), *the inclusion* $(w) \subseteq [g]$ *is equivalent to*

$$\liminf_{t\to 1-0} (1-t)\, g(t)\, w(t) > 0. \tag{4.11}$$

Only the necessity of (4.11) needs further explanation: The inclusion involves, in particular, that $s_n \to 0$ (w) implies $s_n = o(1)\, g(n/(n+1))$. Using the standard argument with the diagonal elements of (w), it follows that

$$(n+1)\Big/ w\left(\frac{n}{n+1}\right) = O(1)\, g\left(\frac{n}{n+1}\right), \quad \text{or} \quad (1-t)\, w(t)\, g(t) \geqslant \epsilon > 0 \quad \left(t = \frac{n}{n+1}\right).$$

We mention below some examples of methods (w) and their relations to certain methods $[g]$:

1. Let $w_\kappa(t) = \kappa(1-t)^{\kappa-1}$, $0 < \kappa \leqslant 1$. Then $(w_\kappa) \approx C_\kappa$ (C_κ denotes the Cesàro method of the order κ). This follows by a Theorem of Miesner [5] from the observation that (w_κ) is essentially the Nörlund mean N_p, $p_n = (n+1)^{\kappa-1}$ (or by [3, Satz 6] and the fact that $(n+1-\nu)^{\kappa-1}/\binom{n-\nu+\kappa-1}{n-\nu} \downarrow$ as $\nu \uparrow$, $\nu \leqslant n$). The function w_κ satisfies (4.10).

2. Let $w_{(\alpha)}(t) = \kappa_\alpha/(1-t)(c_\alpha + \log^\alpha 1/(1-t))$, $\alpha > 1$, $c_\alpha \geqslant (\alpha-1)^{\alpha-1}$, where κ_α is a constant such that $\int_0^1 w_{(\alpha)}(t)\, dt = 1$. A short calculation shows that $w_{(\alpha)}(t) \uparrow$ for $t \uparrow$. If $c_\alpha \geqslant (\alpha-1)^{\alpha-1}\alpha^\alpha$, then (4.10) is satisfied, since with $v = \log 1/(1-t)$, $t(w'_{(\alpha)}(t)/w_{(\alpha)}(t)) = (e^v - 1)(1 - \alpha v^{\alpha-1}/(c_\alpha + v^\alpha))$, and the derivative with respect to v of the last right side, apart from a factor e^v, is

$$1 - \frac{\alpha v^{\alpha-1}}{c_\alpha + v^\alpha} + (1-e^{-v})\left(-\frac{\alpha(\alpha-1)\, v^{\alpha-2}}{c_\alpha + v^\alpha} + \frac{\alpha^2 v^{2\alpha-2}}{(c_\alpha + v^\alpha)^2}\right)$$
$$\geqslant 1 - \frac{\alpha v^{\alpha-1}}{c_\alpha + v^\alpha} + v\left(-\frac{\alpha(\alpha-1)\, v^{\alpha-2}}{c_\alpha + v^\alpha}\right) \geqslant 0.$$

For $c_\alpha = (\alpha-1)^{\alpha-1}\alpha^\alpha$, we denote $(w_{(\alpha)})$ by L_α.

Let $g_\kappa(t) = 1/(1-t)\, w_\kappa(t) = (1/\kappa)(1-t)^{-\kappa}$, $0 < \kappa \leqslant 1$. The method $[g_\kappa]$ will be denoted by $C_\kappa{}^*$. The function g_κ satisfies (1.5), and it satisfies (2.2) and (3.2) when $0 < \kappa < 1$.

Let $g_{(\alpha)}(t) = 1/(1-t)\, w_{(\alpha)}(t)$, $c_\alpha = (\alpha-1)^{\alpha-1}\alpha^\alpha$, $\alpha > 1$. This function satisfies (1.5), and (for $\alpha = 1$) the function $g_{(1)}(t) = 1 + \log 1/(1-t)$ satisfies (2.2) and (3.2). The method $[g_{(\alpha)}]$ will be denoted by $L_\alpha{}^*$.

We have the following relations:

$$\left.\begin{array}{ll} C_\kappa \subseteq C_\kappa{}^*, \quad 0 < \kappa \leqslant 1 & \text{(Theorem 4.3)}, \\ C_\kappa \not\approx C_\kappa{}^*, \quad 0 < \kappa < 1 & \text{(Theorem 4.1)}, \\ C_1 \approx C_1{}^* \quad \text{(since, trivially, } C_1{}^* \subseteq C_1), & \\ C_\kappa{}^* \subseteq C_{\kappa+\epsilon}, \quad \epsilon > 0, \quad 0 < \kappa \leqslant 1 & \text{(Theorem 4.1)}, \end{array}\right\} \tag{4.12}$$

$$\left.\begin{array}{ll} L_\alpha \subseteq L_\alpha^*, \quad \alpha > 1 & \text{(Theorem 4.3)}, \\ L_\alpha^* \subseteq L_{\alpha+\epsilon+1}, \quad \alpha \geqslant 1, \quad \epsilon > 0 & \text{(Theorem 4.1)}, \end{array}\right\} \tag{4.13}$$

$$\left.\begin{array}{ll} L_\alpha^* \subseteq C_\kappa, \quad \alpha \geqslant 1, \quad \kappa > 0 & \text{(Theorem 4.1)}, \\ L_\alpha \subseteq C_\kappa, \quad \alpha > 1, \quad \kappa > 0 & \text{(from (4.13) and } L_\alpha^* \subseteq C_\kappa). \end{array}\right\} \tag{4.14}$$

It follows from (4.14) that $L_\alpha \subseteq \bigcap_{\kappa>0} C_\kappa$ for every $\alpha > 1$. Here the inclusion is strict (note that $L_\alpha \subseteq L_{\alpha+\epsilon+1}$ from (4.13), and that $L_\alpha \not\supseteq L_{\alpha+\epsilon+1}$ since the diagonal terms of these matrices have different order).

Given a Wiener-type method (w), Theorems 1.2 and 4.3 can be used to find conditions for $(w) \subseteq A$. First we have (roughly) $(w) \subseteq [1/(1-t)\,w(t)] \equiv [g]$, and $[g] \subseteq A$, if (1.9) holds. Thus, in case of a triangular A, we have roughly $(w) \subseteq A$ when

$$\sum_{\nu=0}^{n} a_{n\nu} \frac{1}{\left(1 - \frac{\nu}{n+1}\right) w\left(\frac{\nu}{n+1}\right)} = O(1).$$

For w_κ and a monotone ($\nu_n = n$) and triangular A, all the conditions involved are satisfied, and $C_\kappa \subseteq A$, $0 < \kappa \leqslant 1$, *follows from*

$$\sum_{\nu=0}^{n} a_{n\nu} \left(\frac{n+1}{n+1-\nu}\right)^\kappa = O(1). \tag{4.15}$$

5. The Intersection $\cap(w)$, $(w) \supseteq [g]$

The discussion of this problem will be based upon the Remark following Theorem 4.1.

Theorem 5.1. *Let g satisfy* (2.2). *Then a sequence $\{s_\nu\}$ is summable to zero by all methods (w) with $(w) \supseteq [g]$ if and only if*

$$\frac{1}{n+1} \sum_{\nu=0}^{n} s_\nu \to 0 \qquad (n \to \infty) \tag{5.1}$$

and

$$\frac{1}{n+1-m} \sum_{\nu=m}^{n} s_\nu = O\left(1 + g\left(\frac{m}{n+1}\right)\right), \qquad \textit{uniformly for } 0 \leqslant m \leqslant n. \tag{5.2}$$

Proof. We note, first, that (5.1) is necessary since $C_1 \supseteq [g]$. We have $s_n \to 0$ (w) for all $(w) \supseteq [g]$ if and only if $s_n \to 0$ $(Aw - Bw^*)$, whenever

(w), $(w^*) \supseteq [g]$, A, B, $\geqslant 0$, i.e., if and only if

$$\frac{1}{n+1}\sum_{\nu=0}^{n} s_\nu \left(Aw\left(\frac{\nu}{n+1}\right) - Bw^*\left(\frac{\nu}{n+1}\right)\right)$$
$$= \frac{Aw(0)-Bw^*(0)}{n+1}\sum_{\nu=0}^{n} s_\nu + \frac{1}{n+1}\sum_{\nu=1}^{n} s_\nu \int_0^{\nu/(n+1)} \frac{d(AF(x)-BF^*(x))}{(1-x)\,g(x)}$$
$$= I_n + II_n \to 0.$$

Here $I_n \to 0$, and it remains to discuss the condition $II_n \to 0$ when $\phi(x) = AF(x) - BF^*(x)$ is any function in $V[0, 1]$. But

$$II_n = \sum_{\mu=1}^{n} \int_{(\mu-1)/(n+1)}^{\mu/(n+1)} \sum_{\nu=\mu}^{n} s_\nu \frac{d\phi(x)}{(n+1)(1-x)\,g(x)} = \int_0^1 b_n(x)\, d\phi(x),$$

where

$$b_n(x) = \frac{1}{(n+1)(1-x)\,g(x)} \sum_{(n+1)x\leqslant\nu\leqslant n} s_\nu .$$

(The first representation of II_n shows how $\int_0^1 b_n(x)\,d\phi(x)$ must be interpreted when discontinuities of b_n and ϕ coincide.) It follows from (5.1) that

$$b_n(x) \to 0 \quad \text{as} \quad n \to \infty, \qquad \text{for every fixed } x \in [0, 1], \tag{5.3}$$

and from $II_n \to 0$, by the Banach–Steinhaus theorem applied to absolutely continuous functions ϕ, that

$$b_n(x) = O(1) \quad \text{as} \quad n \to \infty, \qquad \text{uniformly for } x \in [0, 1]. \tag{5.4}$$

Conversely, (5.3) and (5.4) imply $II_n \to 0$, and Theorem 5.1 follows since (5.2) is equivalent to (5.4). (Note that $(1-x)\,g(x)\downarrow$.)

One might ask whether (5.1) and (5.2) imply $s_n \to 0$ $[g]$. This is not the case. As a counter-example, let $s_n = g(1 - 1/(n+1))$ for $n = 3^k$, $s_n = 0$ otherwise. We have $s_n \to 0$ (C_1) (note that $s_n = o(n)$, by (2.2)), and this implies (5.2) for $m \leqslant n/2$. But, for $n/2 < m \leqslant n$,

$$\frac{1}{n+1-m}\sum_{\nu=m}^{n} s_\nu \leqslant \frac{g\left(1-\dfrac{1}{n+1}\right)}{n+1-m} = \frac{n+1}{n+1-m}\int_{1-1/(n+1)}^{1} \tilde{g}(t)\,dt$$
$$\leqslant \frac{n+1}{n+1-m}\int_{m/(n+1)}^{1} \tilde{g}(t)\,dt = g\left(\frac{m}{n+1}\right).$$

Hence, $\{s_n\}$ satisfies (5.1) and (5.2), but it is not summable $[g]$ since $s_n \neq o(g(1 - 1/(n+1)))$.

For $g \equiv 1$, Theorem 5.1 shows that a sequence is summable by all methods (w) if and only if it is bounded and summable C_1. Remembering that every Wiener-type method satisfies $(w) \subseteq C_1$, we see that *all methods* (w) *are equivalent for bounded sequences*. In particular, *there is no Wiener-type method which is equivalent to convergence.*

For $g(t) = 1 + \log 1/(1-t)$, our results show that no method (w) is equivalent to $L_1{}^*$, but the Wiener-type methods stronger than $L_1{}^*$ almost exhaust all Fourier-effective (monotone) methods.

6. Nörlund Means and Arithmetical Means

In this section we shall discuss the class of monotone Nörlund means, and the class of monotone arithmetical means, which are stronger than $L_1{}^*$ or some other order summability (different from convergence). For g, the condition

$$g(t) \to \infty \qquad (t \to 1 - 0) \tag{6.1}$$

will be of importance.

THEOREM 6.1. *Let g satisfy* (2.2) *and* (6.1), *and suppose that A is triangular, regular, with $a_{n\nu} \geqslant 0$ $(n, \nu = 0, 1, \ldots)$. Then $A \supseteq [g]$ implies*

$$\inf_{\mu \leqslant \nu \leqslant n} a_{n\nu} \leqslant \frac{M}{(n+1-\mu)\, g\left(\dfrac{\mu}{n+1}\right)} \qquad (0 \leqslant \mu \leqslant n), \tag{6.2}$$

$$\sum_{\nu=\mu}^{n} a_{n\nu} \leqslant M \Big/ \tilde{g}\left(\frac{\mu}{n+1}\right) \qquad (0 \leqslant \mu \leqslant n), \tag{6.3}$$

$$\sum_{\nu \leqslant nt} a_{n\nu} \to 1 \qquad (n \to \infty,\ t \to 1 - 0), \tag{6.4}$$

and the existence of $n_0 \geqslant 0$, $t_0 \in (0, 1)$, $\delta > 0$ such that

$$n \sup_{\nu \leqslant nt} a_{n\nu} \geqslant \delta \quad \text{for} \quad n \leqslant n_0, \qquad t \in (t_0, 1). \tag{6.5}$$

Proof. It follows from (2.2) and (2.4) that

$$M \geqslant \sum_{\nu=\mu}^{n} a_{n\nu}(n+1) \int_{\nu/(n+1)}^{(\nu+1)/(n+1)} \tilde{g}(t)\, dt \geqslant (\inf_{\mu \leqslant \nu \leqslant n} a_{n\nu})(n+1) \int_{\mu/(n+1)}^{1} \tilde{g}(t)\, dt$$

and

$$M \geqslant \sum_{\nu=\mu}^{n} a_{n\nu} \tilde{g}\left(\frac{\nu}{n+1}\right).$$

This gives (6.2) and (6.3). Since, necessarily, $\tilde{g}(t) \to \infty$ $(t \to 1-0)$, it follows that

$$\sum_{nt<\nu\leqslant n} a_{n\nu} \to 0 \qquad (n \to \infty, t \to 1-0)$$

which is equivalent to (6.4). A trivial consequence of (6.4) is (6.5).

In the statement of Theorem 6.1 we have avoided the assumption of monotonicity for A which would have simplified the conditions (but greatly reduced the generality). Condition (6.4) is typical for the inclusion $A \supseteq [g]$ with some (suitable) g. If A is arbitrary, there is nothing much that can be said about possible g's. However, if A is a monotone Nörlund mean, we may always take $[g] = C_\epsilon^*$ (for some $\epsilon > 0$), and if A is a monotone arithmetical mean, we may take $[g] = C_1^*$. To show this is the object of the following theorems. If $[g] = L_1^*$, we obtain relations with Fourier-effectiveness.

Let $\{p_n\}$ be a monotone sequence with $p_n > 0$. For the corresponding Nörlund mean N_p (i.e., the triangular A with $a_{n\nu} = p_{n-\nu}/P_n$, $P_n = p_0 + \cdots + p_n$), condition (6.5) is equivalent to

$$\frac{P_n}{p_n} = O(n). \tag{6.6}$$

THEOREM 6.2. *Let* $\{p_n\}$, $p_n > 0$ $(n \geqslant 0)$, *be monotone, and let the corresponding Nörlund mean* N_p *be regular. Then* (6.6) *is equivalent to each of the following statements*:

$$N_p \supseteq [g] \text{ for some } g \text{ satisfying (2.2) and (6.1)}, \tag{6.7}$$

$$N_p \supseteq L_1^*, \tag{6.8}$$

$$N_p \supseteq C_\epsilon \text{ for some } \epsilon > 0. \tag{6.9}$$

Proof. Condition (6.7), and its special cases (6.8) and (6.9) (see (4.14)) imply (6.6), by Theorem 6.1.

Next, suppose that (6.6) is satisfied, i.e., $p_n/P_n = 1 - P_{n-1}/P_n \geqslant K/(n+1)$ for some $K > 0$. Consequently, $P_{n-1}/P_n \leqslant 1 - K/(n+1)$, and, for $m \leqslant n$,

$$\frac{P_m}{P_n} \leqslant \prod_{\nu=m+1}^{n} \left(1 - \frac{K}{\nu+1}\right) = e^{\sum_{\nu=m+1}^{n} \log(1-K/(\nu+1))} \leqslant e^{-K\sum_{\nu=m+1}^{n} 1/(\nu+1)}$$

$$= O(1)\, e^{-K\log(n+1)/(m+1)} = O(1)\left(\frac{m+1}{n+1}\right)^K.$$

Let $0 < \epsilon < \min(1, K)$. Then $C_\epsilon \subseteq N_p$ follows from (4.15) since

$$\frac{p_{n-\nu}}{P_n} = \frac{p_{n-\nu}}{P_{n-\nu}} \frac{P_{n-\nu}}{P_n} = \frac{O(1)}{n-\nu+1} \left(\frac{n-\nu+1}{n+1}\right)^K$$

implies

$$\sum_{\nu=0}^{n} \frac{p_{n-\nu}}{P_n} \left(\frac{n+1}{n+1-\nu}\right)^\epsilon = O(1).$$

Finally, (6.9) implies (6.7) and (6.8).

Remarks. 1. Hille and Tamarkin [1] have shown, for $0 < p_n \downarrow$, that N_p is F_C-effective if and only if

$$\frac{1}{P_n} \sum_{\nu=0}^{n} P_\nu/(\nu+1) = O(1). \tag{6.10}$$

It follows from

$$\sum_{\nu=0}^{n} \frac{p_{n-\nu}}{P_n} \log \frac{n+1}{n+1-\nu} = \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu \log \frac{n+1}{\nu+1} = \frac{1}{P_n} \sum_{\nu=0}^{n-1} P_\nu \log \frac{\nu+2}{\nu+1}$$

that (6.10) is essentially the condition (1.10) with $g(t) = 1 + \log 1/(1-t)$. Theorem 3.2 then shows that (6.10) is equivalent to $N_p \supseteq L_1{}^*$. Thus, we have a new proof of this result, and (6.10) can be replaced by the simpler condition (6.6) (which immediately implies (6.10)).

2. Karamata [4] has shown that (6.9) is a consequence of (6.10).

There are similar results for arithmetical means M_p (i.e., the triangular A with $a_{n\nu} = p_\nu/P_n$, $P_n = p_0 + \cdots + p_n$). Let $\{p_n\}$ be a sequence with $0 < p_n \uparrow$ for $n \uparrow$. Then (6.5) implies, for some $t \in (0, 1)$,

$$\frac{n}{P_n} \sup_{\nu \leqslant nt} p_\nu \geqslant \delta, \qquad n \geqslant n_0 . \tag{6.11}$$

It follows from (6.11) and $P_n \geqslant \sum_{nt' \leqslant \nu \leqslant n} p_\nu \geqslant n(1-t') \inf_{\nu \geqslant nt'} p_\nu$ that $\inf_{\nu \geqslant nt'} p_\nu \leqslant K \sup_{\nu \leqslant nt} p_\nu$ for some K, depending on $t' \in (0, 1)$, and for all large n. Taking $t' \in (t, 1)$, we see that numbers $\rho > 1$ and $C > 0$ exist such that

$$p_i \leqslant C p_j, \qquad \text{whenever} \quad i \leqslant \rho j. \tag{6.12}$$

Let m be the integer with $n/\rho \leqslant m < n/\rho + 1$. Then it follows from (6.12) that

$$\frac{np_n}{P_n} \leqslant nC \frac{p_m}{p_m + \cdots + p_n} \leqslant nC \frac{p_m}{p_m(n+1-m)},$$

and this shows that

$$\frac{np_n}{P_n} = O(1) \tag{6.13}$$

is a consequence of (6.11) when $p_n \uparrow$.

THEOREM 6.3. *Let* $\{p_n\}$, $p_n > 0$ $(n \geqslant 0)$, *be monotone, and let the corresponding arithmetical mean* M_p *be regular. Then* (6.13) *is equivalent to each of the following statements*:

$$M_p \supseteq [g] \text{ for some } g \text{ satisfying (2.2) and (6.1),} \tag{6.14}$$

$$M_p \supseteq L_1^*, \tag{6.15}$$

$$M_p \supseteq C_1 . \tag{6.16}$$

Proof. When $p_n \downarrow$, then $M_p \supseteq C_1$, and we need only consider the case $p_n \uparrow$. Condition (6.14), and its special cases (6.15) and (6.16) imply (6.13) as in the proof of Theorem 6.2. Suppose now that (6.13) is satisfied. Then (6.16), and even $M_p \approx C_1$, follows (see [2, Satz 16].)

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